

Lecture-8:  
EE1473: Digital Communication Systems

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## Agenda

- Non Uniform Quantization
- Relative Frequency and Probability
- Concept of Conditional Probability, Independence
- Theory of Total Probability
- Bayes' Theorem

## Non-Uniform Quantization

- Usually rms value of signal is much less than the peak value of the signal
- In such cases, we want to give
  - a finer representation of the signal in a “small” amplitude regime and
  - a coarse representation of the signal in “large” amplitude regime
- Two steps in quantization
  - Compress the signal (expand the range of output of the signal in small amplitude regime and shrink that of the signal in large amplitude regime)
  - Pass the compressed signal into the uniform quantizer

## Motivation for Studying Probability Theory

- In digital communications, we make heavy use of statistical tools borrowed from the theories of Probability and Random Variables. Why?
  - Measurements done in experiments are often not precise, rather random due to noise and other uncertainties
  - Thus, an average behavior is often what we can afford to have, or perhaps what we desire to have
  - Tools obtained from the theory will enable us to predict/infer--by calculation--the outcomes of the future/more complex experiments
- Design of communications systems, is done to improve the probabilistic figure of merit, i.e. the probability of making bit errors, frame errors, packet errors, probability of call drops, etc.
- Appendix B: Probability and Random Variable

## Real World Experiments and Mathematical Abstraction

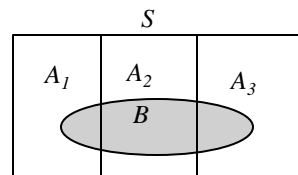
- Experiments
  - Measurement of voltage across a resistance
  - Roll a die
- Three entities in the real world experiments
  - The set of all possible *outcomes*
  - Grouping of the *outcomes* into classes, called *results*
  - The *relative frequencies* of occurrences of the *results*
- The corresponding mathematical abstractions
  - *The sample space*
  - *The set of events*
  - *The probability measure* assigned on each of these *events*

## Fundamental Definitions in Set Theory

- A set is a collection of *objects (elements)*
  - $A = \{v: 0 \leq v \leq 5 \text{ volts}\}$
  - $B_1 = \{1, 2, 3, 4\}$ ,  $B_2 = \{\text{head, tail}\}$
- A subset  $C$  of  $A$  is another set whose elements are also elements of  $A$ 
  - $C = \{1, 2\} \subset B_1$
  - We say  $C$  belongs to  $B_1$
- Set operations: Union and Intersection
  - $B_1 \cup B_2 = \{1, 2, 3, 4, \text{head, tail}\}$
  - $B_1 \cap C = \{1, 2\}$  (Sometimes, a shorthand notation,  $B_1 C$ , is used)
- The *empty set* or *null set*  $\{\}$  (or simply  $\emptyset$ ) is the set having no elements

## Fundamental Definitions in Set Theory

- Two sets  $A$  and  $B$  are *mutually exclusive* or *disjoint* if they have no common elements
  - $A \cap B = AB = ?$
- A partition  $U$  of a set  $S$  is a collection of mutually exclusive subsets  $A_i$  of  $S$  whose union equals  $S$ 
  - $S = A_1 \cup A_2 \cup A_3$  and  $A_i \cap A_j = ?$  for any  $i, j \neq i$
- In the figure below,  $U = [A_1, A_2, A_3]$ , and the subset  $B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B)$



## The *Sample Space* and *Event*

- The sample space,  $\Omega$ , which is called *the certain event*, is a collection of *experimental outcomes* (objects)
  - An object in  $\Omega$  is called a *sample point*, is usually denoted by  $\omega$
- Subsets of the sample space is called *events*
  - Grouping of the *outcomes* into the subsets
  - A set of sample points
  - $A = \{ \omega: \text{some condition on } \omega \text{ is satisfied} \}$ , the event  $A$  is the set of all  $\omega$  such that some condition on  $\omega$  is satisfied
  - An event consisting of a single element is called an *elementary event*

## The *Sample Space* and *Events* (Results)

- Die experiment:  $\Omega = \{1, 2, 3, 4, 5, 6\}$ 
  - $A = \{\omega: \text{odd}\} = \{1, 3, 5\}$
  - $B = \{\omega: \text{even}\} = \{2, 3, 6\}$
- The closed interval of the real line:  
 $\Omega = [0, 1] = \{\omega: 0 \leq \omega \leq 1\}$ 
  - $A = \{\omega: 0.2 \leq \omega \leq 0.7\}$
- All time functions  $f(t)$ ,  $-1 < t < 1$ 
  - An event may be a set of all time functions whose energy is less than 1
- Finite sample space of  $N$  elements  $\rightarrow 2^N$  possible subsets

## *Trial*

- A single performance of an experiment will be called a *trial*
- At each trial we observe a single outcome  $a_i \in S$
- We say an event  $A$  occurs during this trial when  $A$  contains  $a_i$
- From a single trial, a multiple of events can occur
- Roll a die:  $\Omega = \{1, 2, 3, 4, 5, 6\}$ 
  - Now, suppose after a trial, an outcome “1” was observed
  - Then, the events  $\{1\}$ ,  $\{1, 3, 5\}$ ,  $\{1, 3\}$ , and including all the other  $2^5 - 3$  events that contains “1” as an element, it is said, have occurred

## On the Occurrence of Events In a Trial

- We say an event  $A=\{a_1, a_2, a_3\}$  have occurred in a trial, if any one element of the set, namely,  $a_1, a_2,$  or  $a_3,$  was the outcome of the trial
- The event  $\Omega$  occurs in every trials
- Events can be thought of as a question one asks for, to acquire the result of a trial if any of its element has occurred as the outcome of the trial

## Probability Measure

- *An assignment* of a real number in the interval  $[0, 1]$  to the *events* defined on  $\Omega$
- Fair die: All faces occur equally likely with probability  $1/6$
- Unfair die: face-1 event occurs with probability  $1/3,$  the rest 5 faces with  $2/15$
- You can create and use your own rule which suits your need most (your betting rule in Gambling for example)

## Relative Frequency vs. Probability Measure

- The assignment of probability measure to an event  $A$ ,  $P(A)$ , may be done in terms of relative frequency of occurrences in  $N$  independent trials

$$P(A) = \lim_{N \rightarrow \infty} \frac{n_A}{N}$$

where  $n_A$  is the number of occurrence of event  $A$  in  $N$  trials

- Example) a coin is tossed 100 times. The event of head occurred 51 times. Then,  $P(A) = 51/100$
- Example) An experienced gambler watches the cards played, and updates his table of the probability measures on the events of his interests and makes bets accordingly

## Axiomatic Definition of Probability

- The assignment of probability to events should follow the three fundamental rules (Kolmogoroff's axioms)
- 1.  $0 \leq P(A) \leq 1$  (The frequency of an event)
- 2.  $P(\Omega) = 1$  (In every trial there is an outcome)
- 3. If  $A \cap B = \emptyset$ , then  $P(A \cup B) = P(A) + P(B)$ 
  - Die: frequency(1 or 2) = frequency(1) + frequency(2),  
 $\{1\} \cap \{2\} = \emptyset$
- In the theory of probability, all conclusions are direct or indirect consequences of these three axioms
- These conclusions allow us to predict--by calculation--the observable results in the real world experiments

## Examples

- Coin experiment:  $S = \{h, t\}$   
 Its events are the four subsets of  $S$ ,  $\{\emptyset\}$ ,  $\{t\}$ ,  $\{h\}$ ,  $\{h, t\}$ .  
 We may assign  $P\{t\} = p$  and  $P\{h\} = q$ , i.e.,  $p+q=1$ .
  
- Coin toss three times:  $S=\{hhh, hht, hth, htt, ttt, tth, tht, thh\}$ . We may assume the coin is fair and the head and the tail occur with the same probability. In such case, the probability of every elementary events are all the same, and thus it is  $1/8$ .
  - $P\{hht, hhh\} = P\{hht\} + P\{hhh\} = 2/8$

## Compound Experiment and Relative Frequency

- |   | <i>Trials</i> | <i>Results(events)</i> |
|---|---------------|------------------------|
| • An experiment which consists of multiple trials of a simpler experiment | 1             | 0 1                    |
|   | 2             | 1 0                    |
| • Example: a compound experiment of two coin tosses (head=1, tail=0)      | 3             | 1 1                    |
|   | 4             | 0 1                    |
|   | 5             | 0 0                    |
| • Find the frequency of results $\{(1,0)\}$ in 10 trials                  | 6             | 1 1                    |
|   | 7             | 1 0                    |
| 1. $f(1,0) = N(1,0)/10 = 3/10$  | 8             | 0 1                    |
| 2. $f(1,0) = f(\text{first}=1)f(\text{second}=0 \text{first}=1)$          | 9             | 1 0                    |
| $= (5/10) (3/5) = 3/10$   | 10            | 0 1                    |

## Relative Frequency vs. Conditional Probability

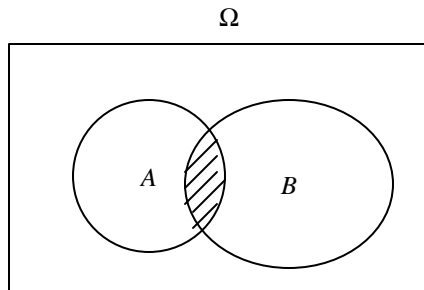
- $f(\text{first}=1, \text{second}=0) = f(\text{first}=1) f(\text{second}=0|\text{first}=1)$   
 $= (5/10) (3/5) = 3/10$
- $f(\text{second}=0|\text{first}=1)$  is the conditional relative frequency of “0” on the second sub-trial given “1” on the first sub-trial
- $f(\text{second}=0|\text{first}=1) = f(\text{first}=1, \text{second}=0)/f(\text{first}=1)$

## Conditional Probability

- Given any two events A and B, the conditional probability  $P(A|B)$  of an event A is defined as
$$P(A | B) := P(AB)/P(B)$$
whenever  $P(B) \neq 0$
- $P(A | A) = 1$
- In the Coin-Toss Three Times experiment, let  $A = \{hhh\}$  and  $B = \{\text{a head in the first toss}\} = \{hhh, hht, hth, htt\}$ 
$$P(A | B) = (1/8)/(1/2) = 1/4$$

## Probability of Joint Event

- Notation:  $P(A, B) \hat{=} P(AB) \hat{=} P(A \cap B)$
- We refer  $P(A, B)$  as the probability of the “joint event A and B.”



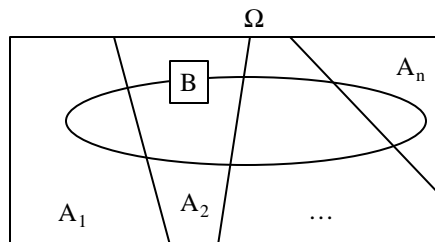
## Conditional Probability

- $P(A, B) = P(A | B) P(B)$   
 $= P(B | A) P(A)$
- A box contains three white balls,  $w_1, w_2,$  and  $w_3$  and two red balls  $r_1$  and  $r_2$ . We remove two balls in succession. What is the probability that the first removed is white and the second is red?
  - $P(\text{first}=\text{white}) = 3/5$
  - $P(\text{second}=\text{red} | \text{first}=\text{white}) = 1/2$
  - $P(\text{first}=\text{white}, \text{second}=\text{red}) = (3/5)(1/2) = 3/10$

## Independence

- If  $P(A|B) = P(A)$  or  $P(B|A) = P(B)$ , the two events A and B are, said to be, (statistically) independent with each other
- Coin Toss Twice:
  - $\Omega = \{hh, ht, th, tt\}$
  - We assigned probability with two numbers  $a$  and  $b$ ,  $a + b = 1$
  - Thus,  $P\{hh\}=a^2$ ,  $P\{ht\}=P\{th\}=ab$ ,  $P\{bb\}=b^2$
  - Note that these probabilities satisfies the axioms:  $a^2+2ab+b^2 = (a+b)^2 = 1$
  - Now, define two events  $A=\{\text{head at the first toss}\}$  and  $B=\{\text{head at the second toss}\}$
  - Note  $P(A)=aa + ab = a$  and  $P(B)=a^2+aa = a$
  - $P(A, B) = P\{hh\} = a^2 = P(A) P(B)$  ) A and B indep.

## Theorem of Total Probability (Very Important)



- If  $U=[A_1, A_2, \dots, A_n]$  is a partition of  $\Omega$  and B is an arbitrary event, then
 
$$P(B) = P(B, A_1) + P(B, A_2) + P(B, A_3) + P(B, A_4)$$

$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) + P(B|A_4)P(A_4)$$

## Bayes' Theorem [Very Important]

- From the results of the conditional probability and the total probability theorem, we could easily get the following,

$$\begin{aligned}
 P(A_i|B) &= \frac{P(A_i, B)}{P(B)} \\
 &= \frac{P(B|A_i) P(A_i)}{\sum_{k=1}^n P(B|A_k) P(A_k)}
 \end{aligned}$$

## Examples of Bayes' Theorem

- Box-1 contains  $a$  white balls and  $b$  black balls. Box-2 contains  $c$  white balls and  $d$  black balls. One ball is drawn from Box-1 and inserted into Box-2. Then, a ball is drawn from the Box-2.
- What is the probability that the ball drawn from Box-2 is white?
  - $A_1 = \{\text{the ball drawn from Box-1} = \text{white}\}$
  - $A_2 = \{\text{the ball drawn from Box-1} = \text{black}\} = A_1^c$
  - $B = \{\text{the ball drawn from Box-2} = \text{white}\}$
  - $P(B) = P(B, A_1) + P(B, A_2) = P(B | A_1) P(A_1) + P(B | A_2) P(A_2)$   
 $= ((c+1)/(c+d+1)) (a/(a+b)) + (c/(c+d+1))*(b/(a+b))$
- What is the probability that the first draw from Box-1 was black, given that a white ball was obtained at the second draw from Box-2  
 $P(A_2|B) = P(B, A_2)/P(B) = P(B|A_2) P(A_2) /P(B)$

## HW#3

- (1)~(3): Couch P3-3, 3-4, 3-8
- (4): If  $A \perp B$ ,  $P(A) = 1/5$  and  $P(B) = 1/2$ , find  $P(A|B)$  and  $P(B|A)$
- (5): A call occurs at random time  $t$  in the interval  $(0, 10)$ .  
Find  $P\{t \leq 5 \mid t \leq 2\}$
- (6): The events  $A$  and  $B$  are mutually exclusive. Can they be independent?
- (7): Suppose  $c$  balls are drawn from a box which contains  $a$  white and  $b$  black balls. What's the probability that at least one white ball is drawn?
- (8): Box-1 contains 100 bulbs of which 10% are bad. Box-2 contains 300 bulbs of which 5% are bad. Two bulbs are selected from a randomly selected box. (a) What is the probability that both are bad. (b) When both are bad, find the probability that they came from Box-1